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Jacobi Summability

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In a paper [6] concerned with symmetric limits of harmonic functions in one and several dimensions, Stein and Zygmund proved the following lemma, a generalization to spheres of the classical theorem of Abel-Stolz.

LEMMA A. *Let* $\lambda > 0$. *If* $\sum a_k P_k^\lambda(1)$ converges to s, then

$$u(r, \theta) = \sum a_k P_k^\lambda(\cos\theta)\, r^k, \qquad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant r < 1,$$

tends to s for $r \to 1$, $\theta = O(1 - r)$. *The conclusion holds if* $\sum a_k P_k^\lambda(1)$ *is summable* (C, α), $\alpha > -1$, *to s.*

Here $P_k^\lambda(x)$ is the ultraspherical polynomial defined by

$$(1 - 2xr + r^2)^{-\lambda} = \sum_{k=0}^{\infty} P_k^\lambda(x)\, r^k.$$

This lemma is used to prove the following theorem on nontangential symmetric limits for harmonic functions in the n ball. Let

$$\sum_{k=0}^{\infty} Y_k(P) \tag{1}$$

be a series of spherical harmonics on the surface of the n-dimensional sphere. The associated harmonic function has the expansion

$$u(x) = u(r, P) = \sum r^k Y_k(P), \tag{2}$$

where

$$r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}, \qquad P = x/|x|.$$

We assume this series converges for $0 \leqslant r < 1$. If P_0 is any point on the

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surface of the sphere, and $r^2 + \rho^2 < 1$, then $\sum (P_0, r, \rho)$ will denote the surface of the $(n - 1)$ dimensional sphere with center rP_0 and radius ρ in the hyperplane perpendicular to the direction determined by P_0.

THEOREM A. *If the series* (1) *converges at the point P_0 to s, then*

$$\frac{\int_{\Sigma(P_0,r,\rho)} u \, d\sigma}{\int_{\Sigma(P_0,r,\rho)} d\sigma},$$

the average value of u over the sphere $\sum (P_0, r, \rho)$, tends to s as $r \to 1$, provided $\rho = O(1 - r)$. The conclusion holds if, instead of convergence, we assume the summability (C, α), $\alpha > -1$, of $\sum Y_k(P_0)$.

Ultimately, similar theorems should be proved for projective spaces, and then the crucial lemma will be an extension of Lemma A to Jacobi series. One technical detail in the Stein–Zygmund proof of Lemma A does not extend to Jacobi series: they use Mehler's integral to estimate the difference of two ultraspherical polynomials, and there is no known simple extension of Mehler's integral to Jacobi polynomials. However, once the problem is generalized to Jacobi series, it is easy to obtain the required estimates and the purpose of this note is to point out that for high dimensions a stronger theorem than that of Stein and Zygmund is true. In Lemma A, if $\lambda > 1$, then the convergence of $\sum a_n P_n^{\lambda}(1) = s$ implies that $u(r, \theta) \to s$ as $r \to 1$, $\theta \to 0$, without the restriction $\theta = O(1 - r)$.

Let $P_n^{(\alpha,\beta)}(x)$ be defined by

$$(1 - x)^{\alpha}(1 + x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1 - x)^{\alpha+n}(1 + x)^{\beta+n}], \alpha, \beta > -1.$$

The analog of Lemma A is

LEMMA 1. *If $\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(1)$* converges to s, then

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(\cos \theta)\, r^n$$

tends to s for $r \to 1$, $\theta = O(1 - r)$. If $\alpha > \frac{1}{2}$ then $u(r, \theta)$ tends to s for $r \to 1$, $\theta \to 0$, without the restriction $\theta = O(1 - r)$.

Let $P_n(\cos \theta; \alpha, \beta) = P_n^{(\alpha,\beta)}(\cos \theta)/P_n^{(\alpha,\beta)}(1)$. A summation by parts shows that it will be sufficient to prove

$$\sum_{n=0}^{\infty} | r^n P_n(\cos \theta; \alpha, \beta) - r^{n+1} P_{n+1}(\cos \theta; \alpha, \beta)| \leqslant C, \qquad 0 \leqslant \theta \leqslant \pi/2. \qquad (3)$$

This will follow from

$$\sum_{n=0}^{\infty} (r^n - r^{n+1}) \mid P_n(\cos\theta;\alpha,\beta)| \leqslant C_1, \qquad 0 \leqslant \theta \leqslant \pi/2, \tag{4}$$

and

$$\sum_{n=0}^{\infty} r^n \mid P_n(\cos\theta;\alpha,\beta) - P_{n+1}(\cos\theta;\alpha,\beta)| \leqslant C_2, \qquad 0 \leqslant \theta \leqslant \pi/2. \tag{5}$$

For $\alpha \geqslant -\frac{1}{2}$ we have

$$\mid P_n(\cos\theta;\alpha,\beta)| \leqslant 1 \qquad [8, (7.32.2)] \tag{6}$$

This proves (4) for $\alpha \geqslant -\frac{1}{2}$. To prove (5) for $\alpha > -1$ we use

$$P_n(\cos\theta;\alpha,\beta) - P_{n+1}(\cos\theta;\alpha,\beta)$$
$$= (1-\cos\theta)\frac{(2n+\alpha+\beta+2)}{2(\alpha+1)} P_n(\cos\theta;\alpha+1,\beta) \qquad [8, (4.5.4)]$$

and so (5) is dominated by

$$\theta^2 A \sum_{n=0}^{\infty} n \mid P_n(\cos\theta;\alpha+1,\beta)| \, r^n = A\theta^2 \left[\sum_{n=0}^{1/\theta} + \sum_{n=1/\theta}^{\infty}\right].$$

In the first sum we again use (6) to get $O(1)$. Observe that we have not used $\theta \leqslant c(1-r)$. In the second sum, we use

$$\mid P_n(\cos\theta;\alpha,\beta)| = O(n^{-\alpha-\frac{1}{2}}\theta^{-\alpha-\frac{1}{2}}), \qquad cn^{-1} \leqslant \theta \leqslant \pi/2 \quad \alpha,\beta > -1 \qquad [8, (7.32.5)] \tag{7}$$

to obtain the estimate for $\alpha \leqslant \frac{1}{2}$. Now,

$$\theta^2 \sum_{n=1/\theta}^{\infty} n^{-\alpha-\frac{1}{2}}\theta^{-\alpha-\frac{3}{2}} r^n \leqslant \theta^{\frac{1}{2}-\alpha}(1-r)^{\alpha-\frac{1}{2}} = O\left(\frac{\theta}{1-r}\right)^{\frac{1}{2}-\alpha} \leqslant A,$$

since $\theta \leqslant c(1-r)$ and $\frac{1}{2}-\alpha \geqslant 0$. If $\alpha > \frac{1}{2}$, then this series is bounded by

$$\theta^{\frac{1}{2}-\alpha} \sum_{n=1/\theta}^{\infty} n^{-\alpha-\frac{1}{2}} = O(\theta^{\frac{1}{2}-\alpha}\theta^{\alpha-\frac{1}{2}}) = O(1)$$

and the condition $\theta = O(1-r)$ is not needed.

To finish the proof we must prove (4) for $-1 < \alpha < -\frac{1}{2}$. Divide the sum at $n = [1/\theta]$ and use

$$P_n^{(\alpha,\beta)}(\cos\theta) = O(n^{\alpha}) \qquad 0 \leqslant \theta \leqslant \frac{1}{n}, \ \alpha < -\tfrac{1}{2} \qquad [8, (7,32.5)] \tag{8}$$

and (7). This gives

$$(1-r)\sum_{n=0}^{1/\theta} r^n + (1-r)\sum_{n=1/\theta}^{\infty} \theta^{-\alpha-\frac{1}{2}} n^{-\alpha-\frac{1}{2}} r^n = O(1) + O(\theta/(1-r))^{-\alpha-\frac{1}{2}} = O(1)$$

since $\alpha < -\frac{1}{2}$ and $\theta \leqslant c(1-r)$.

There are many extensions of Lemma 1. For $\alpha > 0$, the summability method $\sum_{n=0}^{\infty} a_n [P_n^{(\alpha,\beta)}(\cos\theta)/P_n^{(\alpha,\beta)}(1)]^2$ is a regular one, as $\theta \to 0$. So is

$$\sum_{n=0}^{\infty} a_n \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)} \frac{P_n^{(\gamma,\delta)}(\cos\phi)}{P_n^{(\gamma,\delta)}(1)},$$

if $\alpha, \gamma > 0$ and $0 < c \leqslant \theta/\phi \leqslant C < \infty$. There are also connections with Cesàro summable series. For a closely related summability method, Bessel summation, this connection has been studied in detail. See [2] and the references given there. We shall not consider these extensions here. When and if they are needed for applications, they can be worked out using the proofs for Bessel summation as models.

Since $P_n(\cos\theta; \frac{1}{2}, \frac{1}{2}) = [\sin(n+1)\theta]/[(n+1)\sin\theta]$, there is a close relation between these summability methods and classical methods in Fourier series. Riemann summability is $\lim_{\theta\to 0} \sum a_n [P_n(\cos\theta; \frac{1}{2}, \frac{1}{2})]^2$. A natural analog of Riemann summability for Legendre series is

$$\lim_{\theta\to 0} \sum a_n [P_n(\cos\theta; \tfrac{1}{2}, \tfrac{1}{2})]^3.$$

There are many other summability methods related to Fourier series which can be extended to Jacobi series. For example, the Rogosinski method [11, Vol. I, p. 112], which has been so far extended only to Legendre polynomials [9]. Also there are a number of related methods of Szász [7] which can be extended.

Lemma 1 can be applied to Theorem A to eliminate the condition $\rho = O(1-r)$ if $n > 4$ when the series (1) converges. This leads to the seemingly paradoxical condition of having symmetric tangential limits in high dimensional balls for harmonic functions while it is known that symmetric tangential limits need not exist for harmonic functions in two variables which are the restrictions of harmonic functions in several variables. All this really shows is that it is hard for a spherical harmonic expansion in high dimensions to converge. The function must be so smooth that any restriction to two dimensions, while not necessarily harmonic in these two variables, must be smooth enough to have tangential limits.

When restricted to zonal functions on the sphere in n-space, the series (1) becomes

$$f(\theta) = \sum a_n P_n^{(\alpha,\alpha)}(\cos\theta), \qquad \alpha = (n-3)/2, \tag{9}$$

and it is now easy to consider restrictions of this function. This function is expanded in the same type of series for different values of α. This suggests a number of problems of which the most interesting is probably the following: To have uniform convergence of (9) it is sufficient that $f \in \mathrm{lip}(\alpha + \frac{1}{2})$, $\alpha > -\frac{1}{2}$. See Gronwall [3] for Legendre and Laplace series, and Ragozin [4] for the general case. For absolute convergence, $\mathrm{Lip}(\alpha + 1 + t)$ is sufficient. (Actually an L^2 Lipschitz condition is sufficient but that is not relevant here). Now let

$$f(\theta) = \sum a_n P_n^{(\alpha,\alpha)}(\cos\theta), \qquad \alpha \geqslant -\tfrac{1}{2} \text{ fixed},$$

and assume

$$\sum |a_n|\,|P_n^{(\alpha,\alpha)}(\cos\theta)| \leqslant A \sum |a_n|\, n^\alpha < \infty.$$

Expand $f(\theta)$ in a series of $P_n^{(\beta,\beta)}(\cos\theta)$:

$$f(\theta) = \sum b_n P_n^{(\beta,\beta)}(\cos\theta).$$

If $-\frac{1}{2} \leqslant \beta \leqslant \alpha$, this series converges absolutely [5] and, thus, also uniformly. If $\beta > \alpha$, it is possible to choose f so that $\sum b_n P_n^{(\beta,\beta)}(\cos\theta)$ does not converge absolutely. However, it may converge uniformly. Either prove that it always does for some β, $\alpha < \beta \leqslant \alpha + \frac{1}{2}$, or find an example to show that it can fail for all $\beta > \alpha$. One other open question is to show that the condition $\mathrm{Lip}(\alpha + 1)$ is not sufficient for absolute convergence. This is known for the circle, $\alpha = -\frac{1}{2}$, and for Fourier transforms in E^n [10], but the analogous examples for the sphere fail, [1].

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